Finally, it must be pointed out that in this experiment the hydraulic entrance length was about 19 hydraulic diameters. At higher Reynolds number flow than we have considered here, a longer entrance length is needed to achieve fully developed turbulent flow.[3] Furthermore, Eq. (1) is derived by neglecting the magnetic and thermal entrance effects; local heat transfer coefficient measurements are needed in future research to justify this assumption.

References

[1] Murgatroyd, W., "Experiments on Magneto-Hydrodynamic Channel Flow," *Philosophy Magazine*, Vol. 44, No. 359, Dec. 1953, pp. 1348–1354.

[2] Lykoudis, P. S., "Transition from Laminar to Turbulent Flow in Magneto-Fluid Mechanic Channels," *Review of Modern Physics*, Vol. 32, No. 4, Oct. 1960, pp. 796–798.

[3] Brouillette, E. C. and Lykoudis, P. S., "Magneto-Fluid-Mechanic Channel Flow. I. Experiment," *The Physics of Fluids*, Vol. 10, No. 5, May 1967, pp. 995–1001.

[4] Lykoudis, P. S. and Brouillette, E. C., "Magneto-Fluid-Mechanic Channel Flow. II. Theory," *The Physics of Fluids*, Vol. 10, No. 5, May 1967, pp. 1002–1007.

[5] Gardner, R. A., Uherka, K. L., and Lykoudis, P. S., "The Influence of a Transverse Magnetic Field on Forced Convection Liquid Metal Heat Transfer," *AIAA Journal*, Vol. 4, No. 5, May 1966, pp. 848–852.

[6] Krasilnikov, E. Y., "The Effect of a Transverse Magnetic Field upon Convective Heat Transfer in Magnetohydrodynamic Channel Flows," *Magnitnaya Gidrodinamika*, Vol. 1, No. 3, March 1965, pp. 37–40.

[7] Kovner, D. S., Krasilnikov, E. Y., and Danevin, J. G., "Experimental Study of the Effect of a Longitudinal Magnetic Field on Convective Heat Transfer in a Turbulent Tube Flow of Conducting Liquid," *Magnitnaya Gidrodinamika*, Vol. 2, No. 4, April 1966, pp. 101–106.

[8] Krasilnikov, E. Y., "A Study of the Effect of a Magnetic Field on Convective Heat Transfer in a Turbulent Flow of an Electrically Conducting Fluid in Channels," dissertation, Moscow Power Engineering Institute, Moscow, USSR, 1966.

[9] Gardner, R. A. and Lykoudis, P. S., "Magneto-Fluid-Mechanic Pipe Flow in a Transverse Magnetic Field, Part 2. Heat Transfer," *Journal of Fluid Mechanics*, Vol. 48, Pt. 1, Jan. 1971, pp. 129–141.

[10] Hwang, C. L., Knieper, P. J., and Fan, L. T., "Heat Transfer to MHD Flow in the Thermal Entrance Region of a Flat Duct," *International Journal of Heat and Mass Transfer*, Vol. 9, No. 6, June 1971, pp. 773–789.

[11] Jacob, M., *Heat Transfer*, Vol. 2, Wiley, New York, 1963, p. 206.

Three-Dimensional Laminar Boundary Layer over a Body of Revolution at Incidence and With Separation

W. Geissler*

Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt E.V. Aerodynamische Versuchsanstalt Göttingen, F.R. Germany

Nomenclature

x, r = cylindrical coordinates
L = total length of body
ξ, η, ζ = system of rectangular coordinates, oriented with regard to the potential flow
ξ = in direction of streamlines
η = in direction of equipotential lines
ζ = normal to body surface
S = stagnation point of potential flow
U = velocity of potential flow
U_m = meridional component of U
U_ϕ = circumferential component of U
U_∞ = velocity of undisturbed flow
α = angle of incidence of the body

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Index category: Boundary Layers and Convective Heat Transfer—Laminar.

* Research Scientist.

Introduction

The exact prediction of the position of boundary-layer separation for bodies at angle of attack is a problem of great practical importance in aircraft design. Boundary-layer calculations based on one of the common simplifications such as similarity, small crossflow, small perturbation give only approximate solutions of this problem which are not always satisfactory. In the present method, no simplification of the boundary-layer equations is introduced. The complete three-dimensional laminar boundary-layer equations are solved numerically by an implicit finite difference technique.

Recently, Blottner and Ellis[1] and Wang[2] also presented methods based on finite difference concepts. A discussion of separation patterns over inclined bodies of revolution was given by Wang.[3] Both authors use coordinate systems which are fixed to the body geometry. To extend the calculation over the entire body surface a separate quasi two-dimensional calculation process along one line of symmetry is necessary to specify initial boundary-layer profiles along this line. The application of both methods is limited to simple bodies of revolution like prolate spheroids or elliptic-paraboloids where the inviscid flow data can be obtained analytically.

In the present method a streamline coordinate system is introduced. The coordinates are represented by the streamlines and potential lines of the outer inviscid flow. Using streamline coordinates the extension of the numerical boundary-layer calculation over the entire body surface is a straightforward process. No transformation from one into another coordinate system is necessary. A finite difference scheme is chosen which permits one to include the lines of symmetry of the flow into the calculation process. No initial data along the lines of symmetry have to be specified. The outer inviscid flow is calculated numerically by a singularity method which has been developed by the author.[4] Therefore this method is applicable to bodies of revolution with arbitrary cross-sections. To determine the position of boundary-layer separation the condition of numerical stability (Courant-Friedrich-Levy condition) serves as a useful criterion.

Governing Equations

Figure 1 describes the problem and gives the notation in a streamline coordinate system: the rectilinear coordinate system ξ, η, ζ is oriented in the direction of streamlines, equipotential lines, and normal to the surface, respectively. The corresponding velocity components in these directions are, u, v, w, whereas U is the velocity of the outer inviscid flow, U_∞ is the undisturbed mainflow velocity, and α the angle of attack. Here the velocity component u, which is parallel to the streamlines of the outer flow, is the primary velocity and the velocity component v normal to the streamlines gives the secondary flow in the boundary layer. All lengths are made dimensionless with the

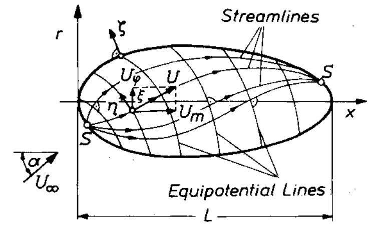

Fig. 1 Flow around a body of revolution at incidence.

total body length L, all velocities with U_∞, and pressures with $\rho U_\infty{}^2$, where ρ is the density. The quantities ζ and w are multiplied in addition by $Re^{1/2}$, where Re is the Reynolds number ($Re = U_\infty L/\nu$) and ν the kinematic viscosity.

The boundary-layer equations for steady three-dimensional laminar incompressible flow in curvilinear coordinates in dimensionless form are:

Equation of continuity

$$\frac{\partial u}{h_1 \partial \xi} + \frac{\partial v}{h_2 \partial \eta} + \frac{\partial w}{\partial \zeta} - K_1 u - K_2 v = 0 \tag{1}$$

Equation of momentum in ξ direction

$$u\frac{\partial u}{h_1 \partial \xi} + v\frac{\partial u}{h_2 \partial \eta} + w\frac{\partial u}{\partial \zeta} - K_2 uv + K_1 v^2 = -\frac{\partial p}{h_1 \partial \xi} + \frac{\partial^2 u}{\partial \zeta^2} \tag{2}$$

Equation of momentum in η direction

$$u\frac{\partial v}{h_1 \partial \xi} + v\frac{\partial v}{h_2 \partial \eta} + w\frac{\partial v}{\partial \zeta} - K_1 uv + K_2 u^2 = -\frac{\partial p}{h_2 \partial \eta} + \frac{\partial^2 v}{\partial \zeta^2} \tag{3a}$$

The boundary conditions are:

$$\begin{aligned} u = v = w = 0 \quad &\text{at} \quad \zeta = 0 \\ u = U, v = w = 0 \quad &\text{as} \quad \zeta \to \infty \end{aligned} \tag{3b}$$

The local curvature parameters K_1, K_2 are defined as follows:

$$K_1 = -\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}; \qquad K_2 = -\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta} \tag{4}$$

In Eqs. (1–4) h_1 and h_2 are the metric coefficients such that $h_1\, d\xi$ is the actual differential length along streamlines and $h_2\, d\eta$ is the actual differential length along equipotential lines. In streamline coordinates the term h_1 is defined by:

$$h_1 = 1/U \tag{5}$$

Method of Solution

The differential Eqs. (1–3b) represent a mixed initial value and boundary value problem. Before the actual numerical boundary-layer calculation can start initial boundary-layer profiles near the front stagnation point as well as potential flow data at the outer edge of the boundary layer have to be specified. For the computation of initial boundary-layer profiles along equipotential lines near the front stagnation point the well-known three-dimensional stagnation point boundary-layer solution of Howarth[5] is used.

The potential velocity distribution on the body surface is calculated by a singularity method[4] giving the flow quantities in a large number of discrete surface points. It is possible to calculate the potential velocity components in arbitrary surface points by means of a spline-interpolation process.

Knowing the surface velocity components in meridional direction (U_m) and in circumferential direction (U_ϕ) (Fig. 1), one obtains the surface streamline equation in the form:

$$U_\phi/U_m = r\, d\phi/ds \tag{6}$$

where r is the radius of the body at station x, ϕ is the circumference angle, and ds is the arc length along meridians. Equation (6) is integrated numerically by the Runge-Kutta method. The velocity potential Φ is determined by

$$\Phi = \int_{\xi_0}^{\xi} U h_1\, d\xi \tag{7}$$

with integration along streamlines. The equipotential lines are simply defined by the condition Φ = constant. With Eqs. (6) and (7) a complete set of streamline coordinates can be calculated. But the mesh sizes formed by the stream- and equipotential lines may not be arbitrary with respect to the three-dimensional finite difference method. Hall[6] has extended the implicit finite difference procedure of the Crank-Nicholson type to the three-dimensional case. In contrast to the two-dimensional case, this method is not always stable. A numerical stability condition, namely the Courant-Friedrich-Levy condition, must be satisfied through the boundary layer.

For the finite difference scheme investigated by Hall, which is used in the present method, the stability condition takes the form:

$$\left| \frac{v}{u}\frac{h_1 \Delta\xi}{h_2 \Delta\eta} \right| \leq 1 \tag{8}$$

This condition is based on the concept of the zone of dependence. If the ratio of mesh sizes, $h_1\Delta\xi/(h_2\Delta\eta)$, in Eq. (8) is equal to one or is smaller than unity, the mesh sizes are favorable with respect to the stability condition.

A mesh of streamlines and equipotential lines with such controlled mesh sizes is plotted in Fig. 2 for an ellipsoid of revolution of axis ratio $a/b = 4$ at angle of attack $\alpha = 15°$. The streamlines have to be reorientated several times along equipotential lines. Flow data are transformed from one system into the neighboring one by a simple interpolation process along the corresponding equipotential lines.

With known potential flow in all netpoints of the coordinate system and with initial boundary-layer profiles in the vicinity of the stagnation point, the numerical solution of Eqs. (1–3b) can start. The differential quotients are approximated in the usual way by finite differences around the central point of the difference scheme. The method is implicit in ζ direction and involves the solution of simultaneous linear algebraic equations with tridiagonal coefficient matrices. In correspondence with Hall's method, the three boundary-layer equations are solved by an iteration process.

The computation marches from netpoint to netpoint along equipotential lines and streamlines including both lines of symmetry.

In Fig. 3 an example of the boundary-layer calculation is given for the same ellipsoid as in Fig. 2 and for the angle of incidence

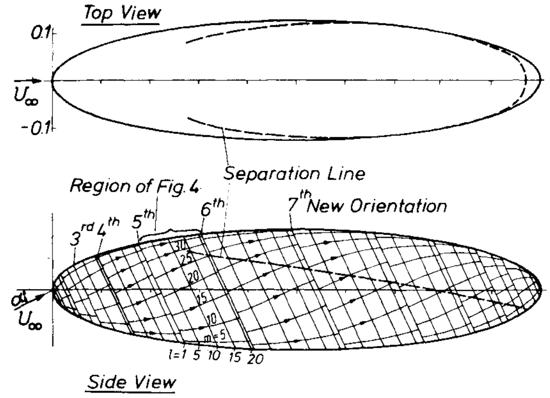

Fig. 2 Streamline net with separation line for an ellipsoid of revolution of axis ratio $a/b = 4$ at $\alpha = 15°$ angle of attack, l = number of equipotential line, m = number of streamline.

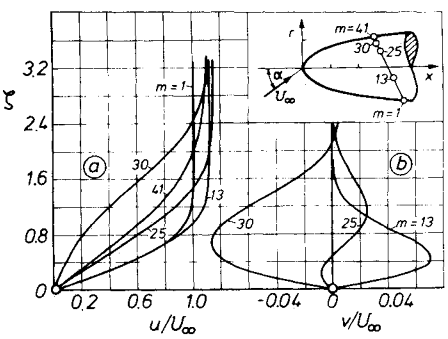

Fig. 3 Velocity distribution of the three-dimensional boundary layer on an ellipsoid of revolution of axis ratio, $a/b = 4$, at angle of attack $\alpha = 15°$; a) Primary flow profiles, u/U_∞, in direction of outer flow streamlines, b) Secondary flow profiles, v/U_∞, normal to the direction of outer flow streamlines. The profiles are given for potential line $l = 13$ of Fig. 2 at streamline stations m from Table 1.

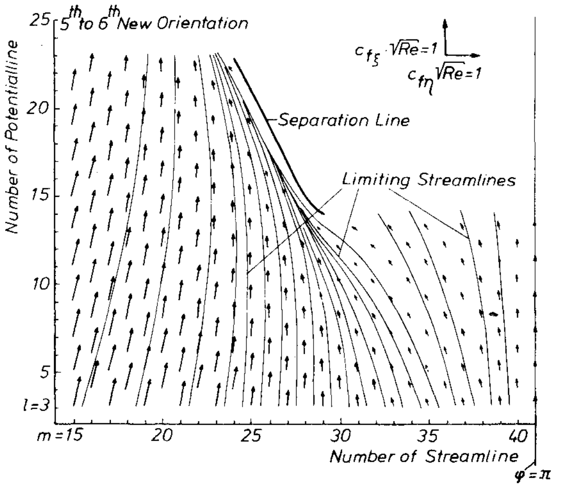

Fig. 4 Vectors of the local skin friction coefficient c_f and limiting streamlines. Beginning of separation between the 5th and 6th new orientation in Fig. 2, that is between $x/L = 0.251$ and $x/L = 0.385$ along the windward symmetry line.

$\alpha = 15°$. Both the profiles of the primary velocity, u, and of the secondary velocity, v, are plotted against the dimensionless distance ζ from the wall for five and three different streamlines, respectively, the numbers of which are given in Fig. 2. The area covered is close to the separation line of Fig. 2. In the region of maximum negative secondary flow (point $m = 30$ in Fig. 3) the u-profiles have a point of inflection corresponding to a minimum of wall shearing stress in this direction.

Separation

The numerical boundary-layer calculation can be extended over the body surface until a point is reached, where the stability condition Eq. (8) is violated. Figure 4 represents an example of the numerical calculation again for the ellipsoid of axis ratio $a/b = 4$ at an angle of attack of $\alpha = 15°$ (Fig. 2). In a small region which is marked in Fig. 2 the distribution of the vector of the local skin friction coefficient c_f is plotted as defined by

$$\tfrac{1}{2}c_f = \frac{\tau_o}{\rho U_\infty{}^2} = \frac{1}{(Re)^{1/2}}\left[\left(\frac{\partial u}{\partial \zeta}\right)^2 + \left(\frac{\partial v}{\partial \zeta}\right)^2\right]^{1/2}_{\zeta\to 0} \tag{9}$$

with τ_o as shearing stress at the wall.

The limiting streamlines are calculated by integration of the directions of the c_f-components. These wall streamlines run into one line forming an envelope. Near this envelope a maximum of reversed secondary flow in connection with a decrease of the primary velocity components occurs (see Fig. 3). The boundary-layer thickness reaches a steep maximum in this region. The computation can be extended downstream of the first numerical instability point always marching along equipotential lines until numerical instability occurs. The so determined instability line corresponds to the envelope of the limiting streamlines and is plotted in Fig. 2. This line is interpreted as the separation line of the free vortex layer type in the sense of Maskell[7] and Wang.[3] The separation line ends at the windward symmetry line of the potential flow, where reversed flow occurs in the boundary layer.

Table 1 Streamline station coordinates

m	x/L	$\phi[°]$
1	0.360	0
13	0.322	71
25	0.277	122
30	0.264	141
41	0.254	180

Similar results have been obtained for numerous bodies of revolution with different axis ratios and angles of attack.[8] The present numerical method has also been tested for the simple cases of a sphere and a semi-infinite cylinder in cross flow. In both cases very satisfactory results with respect to the location of the separation point or separation line have been obtained with analytical methods.

References

[1] Blottner, F. G. and Ellis, M. A., "Finite-Difference Solution of the Incompressible Three-Dimensional Boundary Layer Equations for a Blunt Body," *Computers and Fluids*, Vol. 1, Pergamon, New York, 1973, pp. 133–158.

[2] Wang, K. C., "Three-Dimensional Boundary Layer near the Plane of Symmetry of a Spheroid at Incidence," *Journal of Fluid Mechanics*, Vol. 43, Pt. 1, Aug. 1970, pp. 187–209.

[3] Wang, K. C., "Separation Patterns of Boundary Layer over an Inclined Body of Revolution," *AIAA Journal*, Vol. 10, No. 8, Aug. 1972, pp. 1044–1050.

[4] Geissler, W., "Berechnung der Potentialströmung um rotationssymmetrische Rümpfe, Ringprofile und Triebwerkseinläufe," *Zeitschrift für Flugwissenschaften*, Bd. 20, 1972, pp. 457–462.

[5] Howarth, L., "The Boundary Layer in Three-Dimensional Flow, Pt. 2: The Flow near a Stagnation Point," *Philosophical Magazine*, Vol. 42, Ser. 7, No. 335, Dec. 1951, pp. 1433–1440.

[6] Hall, M. G., "A Numerical Method for Calculating Steady Three-Dimensional Laminar Boundary Layers," RAE TR 67145, Royal Aircraft Establishment, Farnborough, England, June 1967, pp. 1–23.

[7] Maskell, E. C., "Flow Separation in Three Dimensions," RAE Rept. Aero 2565, Royal Aircraft Establishment, Farnborough, England, 1955.

[8] Geissler, W., "Berechnung der Dreidimensionalen Laminaren Grenzschicht an Schrägangeströmten Rotationskörpern mit Ablösung," *Ingenieur Archiv*, to be published.

Penny-Shaped Crack in a Linear Viscoelastic Medium under Torsion

Purna N. Kaloni* and Raymond Smith†

University of Windsor, Windsor, Ontario, Canada

I. Introduction

It is well known that to solve a viscoelastic boundary value problem that involves the boundary conditions of mixed type, the classical correspondence principle of obtaining the viscoelastic solution from the associated elastic solution is, in general, no longer applicable. Graham[1] has proposed a correspondence principle for problems with time-dependent boundary regions and has applied it[2] to solve the problems of a penny-shaped crack subjected to a tension normal to the plane of the crack. Ting[3] has developed a technique of solving problems with moving boundaries and has applied it to obtain the contact stresses between an axisymmetric rigid identer and a viscoelastic half space. Present authors[4] have applied this method to solve the problems of a penny-shaped crack in a viscoelastic medium under shear.

In the present Note we solve the related problem of a penny-shaped crack in a linear viscoelastic medium under torsion. Expressions for the stress distribution in the plane of the crack, displacement over the surface of the crack, and the stress intensity factor are given in the closed form. These quantities

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* Professor of Applied Mathematics, Department of Mathematics.

† Graduate Student.